Prolonged decay and CP asymmetry

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The time evolution of unstable particles that occur in the expanding universe is investigated. The off-shell effect not included in the Boltzmann-like equation is important for the decay process when the temperature becomes much below the mass of unstable particle. When the off-shell effect is taken into account, the thermal abundance of unstable particles at low temperatures has a power law behavior of temperature T, $(\Gamma/M)(T/M)^{\alpha+1}$ unlike the Boltzmann suppressed $e^{-M/T}$, with the power α related to the spectral rise near the threshold of the decay and with Γ the decay rate. Moreover, the relaxation time towards the thermal value is not governed by the exponential law; instead, it is the power law of time. The evolution equation for the occupation number and the number density of the unstable particle is derived, when both of these effects, along with the cosmic expansion, are included. We also critically examine how the scattering of thermal particles may affect the off-shell effect to the unstable particle. As an application showing the importance of the off-shell effect we compute the time evolution of the baryon asymmetry generated by the heavy X boson decay. It is shown that the out-of equilibrium kinematics previously discussed is changed; this change becomes considerable for large values of $\Gamma/H \gg 1$ where H is the Hubble rate at the temperature equal to the X-boson mass, while we confirm the previous result for small values of $\Gamma/H \ll 1$. [S0556-2821(98)02516-8]

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I. INTRODUCTION

There are many short-lived particles that have existed in abundance in the early universe whose temporary presence did not leave behind any measurable effect. Important exceptions to this exist, such as the neutron which certainly is the key for the explanation of the element abundance of the present universe.

A theoretical estimate of the abundance of these unstable particles after the cosmic temperature drops below the mass of the unstable particle is very important for subsequent time evolution. Most works in the past [1] were based on the Boltzmann equation that takes into account relevant reactions in the expanding universe. The use of the Boltzmann equation has however been questioned recently [2]; a more precise quantum mechanical description of the decay process in a thermal medium should contain important off-shell contributions not properly treated in the Boltzmann approach. These off-shell effects are eminent in the low temperature region. Low temperature effects are clearly important in this problem, since unstable particles are typically very nonrelativistic when they disappear in the early universe.

In the present work we shall develop a general formalism of computing the time evolution of the net number density of unstable particles and clarify the off-shell effect. The off-shell effect appears in two ways: first, in a slower relaxation towards the equilibrium abundance and second, in a larger equilibrium value not suppressed by the Boltzmann factor such as $e^{-\Delta M/T}$ where ΔM is the mass difference of the parent and the daughter particles. It is shown below that the off-shell effect becomes dominant below some temperature $T_{\rm eq}$. The abundance of unstable particles then follows the power law; $n/T^3 \approx (\Gamma/M)(T/M)^{\alpha+1}$, where α is a parameter related to the threshold behavior of the spectral function for the decay and Γ is the decay rate. Thus, unstable particles do not disappear suddenly. Instead, their abundance gradually decreases with a power of decreasing temperature as the universe expands. Physical processes that follow after the decay are then prolonged. The off-shell effect turns out to be more prominent for a larger decay rate.

We next consider as an illustrative application of this general result the hypothetical X boson decay that may have created the matter-antimatter asymmetry when they decay [3,4]. We find that the time evolution of the baryon asymmetry is substantially changed and the severe lower bound of the X boson mass is considerably relaxed by the off-shell effect. For the first time we find that some mode of the X boson decay for baryogenesis is excluded due to the off-shell effect. This is the S-wave decay mode into a boson-pair.

This paper is organized as follows. In Sec. II the theoretical model of unstable particle decay is explained. This is a field theoretical extention of the harmonic model for the quantum dissipation in thermal medium discussed in [2]. We first present and formally solve the quantum mechanical model of the decay of excited levels in thermal medium. A great virture of this model is that its integrability leads to explicit formulas for many quantities of interest. One can clearly see how the off-shell effect arises in these formulas. Extention to the unstable particle decay in field theory models can be made, but it is in general complicated and not readily solvable. But fortunately, in a thermal medium far away from the degeneracy limit which is relevant in the early universe the decay process is approximately described by this class of solvable quantum mechanical models extended to infinitely many decay channels. In Sec. III the occupation number and the number density of a species of unstable particles is calculated and its time evolution equation is derived in the expanding universe. The stationary abundance when the cosmic expansion is switched off is worked out, and its behavior at both high and low temperatures is studied in detail. In Sec. IV we pay special attention to the off-shell effect and its role in cosmology. We also discuss a possible effect of the incoherence due to the scattering off thermal particles and its role to the decay process in thermal medium. In Sec. V we apply previous results to the problem of baryogenesis. The time evolution equation for the baryon asymmetry is derived, including the off-shell effect. This equation is analyzed both analytically and numerically, and a comparison is made when only the on-shell contribution is retained.

II. MODEL OF UNSTABLE PARTICLE DECAY

We first present an exactly solvable model of the decay of excited levels in a thermal medium, an extension being made to the case of many decay channels. This is a slight extention of our previous harmonic model [2]. We then explain how the two-particle decay of unstable particles in any quantum field theory can be approximately described by this class of quantum mechanical models of infinitely many decay channels. The approximation is valid for the thermal medium of the low occupation number, the circumstance far away from the degeneracy limit.

We assume that an excited state $|1\rangle$ of energy E_1 is given by applying a creation operator to the vacuum $|0\rangle$; $|1\rangle = c^\dagger|0\rangle$. There are continuously many states degenerate with this, $b^\dagger(\omega)|0\rangle$ as decay states. Here ω is the energy of the continuously many levels. Without specifying the nature of the decay process, one may take the Hamiltonian that governs the decay as

$$H = E_1 c^\dagger c + \int_{\omega_c}^{\infty} d\omega\ \omega b^\dagger(\omega) b(\omega) + \int_{\omega_c}^{\infty} d\omega \sqrt{\sigma(\omega)} (b^\dagger(\omega) c + c^\dagger b(\omega)). \tag{2.1}$$

Here ω_c is the threshold of the continuous states taken to be $\omega_c < E_1$, and $\sigma(\omega)$ characterizes the decay interaction. This Hamiltonian is a general one with regard to the decay process, complications being hidden in identification of the composite operator $b^\dagger(\omega)$ and the spectral form of interaction $\sigma(\omega)$.

As emphasized elsewhere [5], the dynamical system thus specified is exactly solvable; one may explicitly construct the diagonal operator $B^\dagger(\omega)$ and the eigenstate $|\omega\rangle_S = B^\dagger(\omega)|0\rangle$ that diagonalizes the decay Hamiltonian:

$$B^\dagger(\omega) = b^\dagger(\omega) + F(\omega + i0^+) \left(-\sqrt{\sigma(\omega)} c^\dagger + \int_{\omega_c}^{\infty} d\omega \frac{\sqrt{\sigma(\omega)\sigma(\omega')}}{\omega' - \omega - i0^+} b^\dagger(\omega') \right). \tag{2.2}$$

Here $F(z)$ is analytic except on the branch cut along the real axis $z > \omega_c$ and is given by

$$F(z) = \frac{1}{-z + E_1 - \int_{\omega_c}^{\infty} d\omega \frac{\sigma(\omega)}{\omega - z}}. \tag{2.3}$$

One can explicitly check that the canonical commutation relation

$$[B(\omega),\ B^\dagger(\omega')] = \delta(\omega - \omega'), \tag{2.4}$$

and the important inversion relation and the Hamiltonian equivalence

$$b^\dagger(\omega) = B^\dagger(\omega) + \int_{\omega_c}^{\infty} d\omega' \times \frac{\sqrt{\sigma(\omega)\sigma(\omega')} F^*(\omega' + i0^+)}{\omega - \omega' + i0^+} B^\dagger(\omega'), \tag{2.5}$$

$$c^\dagger = -\int_{\omega_c}^{\infty} d\omega \sqrt{\sigma(\omega)} F^*(\omega + i0^+) B^\dagger(\omega), \tag{2.6}$$

$$H = \int_{\omega_c}^{\infty} d\omega\ \omega B^\dagger(\omega) B(\omega). \tag{2.7}$$

The basic reason of integrability is saturation of the unitarity relation by an ''elastic'' one:

$$F(\omega + i0^+) - F(\omega - i0^+) = 2\pi i\ \sigma(\omega) |F(\omega + i0^+)|^2 \equiv 2\pi i\ H(\omega). \tag{2.8}$$

The quantity $H(\omega)$ is characterized as the overlap between the prepared state $c^\dagger|0\rangle$ and the eigenstate $|\omega\rangle_S$:

$$H(\omega) = |\langle 0|c|\omega\rangle_S|^2. \tag{2.9}$$

In the weak coupling of $\sigma(\omega) \ll M$, the spectral function $H(\omega)$ has a Breit-Wigner form as seen from the formula

$$H(\omega) = \frac{\sigma(\omega)}{(\omega - E_1 + \Pi(\omega))^2 + (\pi\sigma(\omega))^2}, \tag{2.10}$$

where $\Pi(\omega)$ is real and given by the dispersion integral

$$\Pi(z) = \mathcal{P} \int_{\omega_c}^{\infty} d\omega \frac{\sigma(\omega)}{\omega - z}. \tag{2.11}$$

As is well known, there is a simple pole of $F(z)$ in the second Riemann sheet near the real axis which describes the time evolution in the form of the exponential decay; $e^{-iE_1 t - \Gamma t/2}$. The imaginary part of this pole coincides, in the weak coupling limit, with the decay rate given by perturbation theory: $\Gamma = 2\pi\sigma(E_1)$.

Since the operator solution is known, one can explicitly write down many quantities of interest; for instance the nondecay amplitude of a pure unstable state $c^\dagger|0\rangle$ is given by

$$\langle 1|e^{-iHt}|1\rangle=\langle 0|c\ e^{-iHt}\ c^{\dagger}|0\rangle$$
$$=\int_{\omega_c}^{\infty}d\omega\ \sigma(\omega)|F(\omega+i0^{+})|^{2}e^{-i\omega t}. \tag{2.12}$$

We are primarily interested in the occupation number and its time evolution in the cosmological thermal medium. Let us first recall this quantity in the pure state $|\psi(t)\rangle=e^{-iHt}|1\rangle$ at time t,

$$\langle\psi(t)|c^{\dagger}c|\psi(t)\rangle=\langle 1|e^{iHt}c^{\dagger}ce^{-iHt}|1\rangle=\langle 1|c^{\dagger}(t)c(t)|1\rangle. \tag{2.13}$$

Here $c(t)=e^{iHt}ce^{-iHt}$ is the Heisenberg operator and in this model

$$c^{\dagger}(t)=-\int_{\omega_c}^{\infty}d\omega\sqrt{\sigma(\omega)}F^{*}(\omega+i0^{+})e^{i\omega t}\ B^{\dagger}(\omega). \tag{2.14}$$

More conveniently, it is written in terms of the parent and daughter operators,

$$c^{\dagger}(t)=g(t)c^{\dagger}+i\int_{\omega_c}^{\infty}d\omega\sqrt{\sigma(\omega)}h(\omega,t)e^{i\omega t}b^{\dagger}(\omega), \tag{2.15}$$

$$g(t)=\int_{\omega_c}^{\infty}d\omega\ H(\omega)e^{i\omega t}=\int_{\omega_c}^{\infty}d\omega|\langle 1|\omega\rangle_{S}|^{2}e^{i\omega t}, \tag{2.16}$$

$$h(\omega,t)=i\ F^{*}(\omega+i0^{+})-ik(\omega,t),$$
$$\dot{k}(\omega,t)=i\ e^{-i\omega t}g(t), \tag{2.17}$$

where a condition $k(\omega,\infty)=0$ is imposed such that the asymptotic value is

$$h(\omega,\infty)=iF^{*}(\omega+i0^{+}). \tag{2.18}$$

The occupation number in the pure state is thus given by

$$\langle 1|c^{\dagger}(t)c(t)|1\rangle=|g(t)|^{2}. \tag{2.19}$$

There is a very useful way [2] to compute the basic function $g(t)$. One can use the analytic property of $F(z)$ to express this function as a sum of two contour integrals as shown in Fig. 1; the first one encircling the pole in the second sheet (C_0) and the second continuous integral along a complex path C_1;

$$g(t)=\frac{1}{2\pi i}\left(\int_{C_0}+\int_{C_1}\right)dz\ F(z)e^{izt}\equiv g_{0}(t)+g_{1}(t). \tag{2.20}$$

Physically, this second continuous contribution $g_1(t)$ gives the off-shell effect, while the pole contribution $g_0(t)$ essentially gives the on-shell effect. Both terms decrease as $t\to\infty$, but the C_1 integral has a power dependence [5] in contrast to the exponential form of the pole term; $g_1(t)\propto t^{-\alpha-1}$, where α is related to the threshold behavior of the spectral function, $\sigma(\omega)\approx c(\omega-\omega_c)^{\alpha}$, near $\omega=\omega_c$.

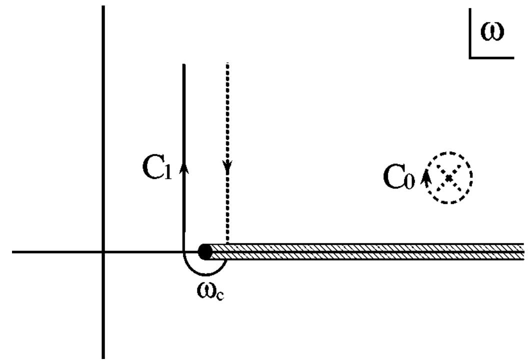

FIG. 1. Contour for the ω integral that separates the on-shell contribution (C_0) from the off-shell one (C_1). The dashed parts are in the second Riemann sheet continued via the cut starting from the threshold ω_c.

There is an equivalent and more intuitive way to separate the on-shell and the off-shell contributions. For this we go back to the ω integral along the real axis,

$$g(t)=\int_{\omega_c}^{\infty}d\omega\frac{\sigma(\omega)}{(\omega-E_1+\Pi(\omega))^{2}+(\pi\sigma(\omega))^{2}}e^{i\omega t}. \tag{2.21}$$

Assuming the weak coupling, $\Pi(\omega)\ll E_1$, one separates the region of integration into the two parts, one around the pole, $\omega\approx E_1-\Pi(E_1)$, and the rest of the region which is dominated near the threshold, $\omega=\omega_c$, for a large time. This approximation gives

$$g(t)\ \approx\ \exp[iE_1t-\pi\sigma(E_1)t]+\frac{1}{E_1^{2}}\int_{\omega_c}^{E_c}d\omega\ \sigma(\omega)e^{i\omega t}, \tag{2.22}$$

where E_c is a physical cutoff scale of order E_1. The first term is the on-shell contribution, while the second is the off-shell contribution. In computation of physical quantities in thermal medium there are other energy scales such as the temperature T, which may replace this cutoff by the factor $e^{-\omega/T}$. The important message is that the off-shell effect at late times is determined by the ω integration near the threshold region.

Extention to the many channel problem is straightforward. We denote the channel by an index i and write the decay interaction as

$$\int_{\omega_c}^{\infty}d\omega\sum_{i}\ \sqrt{\sigma_i(\omega)}(b_i^{\dagger}(\omega)c+c^{\dagger}b_i(\omega)). \tag{2.23}$$

Only the key formulas are quoted:

$$B_i^\dagger(\omega)=b_i^\dagger(\omega)+F(\omega+i0^+)\Bigg(-\sqrt{\sigma_i(\omega)}c^\dagger + \int_{\omega_c}^\infty d\omega'\sum_k \frac{\sqrt{\sigma_i(\omega)\sigma_k(\omega')}}{\omega'-\omega-i0^+}b_k^\dagger(\omega')\Bigg), \tag{2.24}$$

$$F(z)=\frac{1}{-z+E_1-\int_{\omega_c}^\infty d\omega \frac{\sum_i \sigma_i(\omega)}{\omega-z}}, \tag{2.25}$$

$$c^\dagger(t)=g(t)c^\dagger+i\int_{\omega_c}^\infty d\omega\; h(\omega,t)e^{i\omega t}\sum_i \sqrt{\sigma_i(\omega)}b_i^\dagger(\omega), \tag{2.26}$$

$$\begin{aligned} b_i^\dagger(\omega,t)&=e^{iHt}b_i^\dagger e^{-iHt} \\ &=e^{i\omega t}\,(\sqrt{\sigma_i(\omega)}\; ih(\omega,t)c^\dagger+b_i^\dagger(\omega)) \\ &\quad+\sqrt{\sigma_i(\omega)}\int_{\omega_c}^\infty d\omega' \frac{\sum_k \sqrt{\sigma_k(\omega')}b_k^\dagger(\omega')}{\omega-\omega'+i0^+} \\ &\quad\times(ih(\omega,t)e^{i\omega t}-ih(\omega',t)e^{i\omega' t}), \end{aligned} \tag{2.27}$$

$$g(t)=\int_{\omega_c}^\infty d\omega \sum_i \sigma_i(\omega)|F(\omega+i0^+)|^2e^{i\omega t}, \tag{2.28}$$

$$h(\omega,t)=i(F^*(\omega+i0^+)-k(\omega,t)), \tag{2.29}$$

$$k(\omega,t)=\frac{1}{2\pi i}\int_{C_0+C_1} dz \frac{F(z)}{z-\omega}e^{i(z-\omega)t}. \tag{2.30}$$

Note that both the analytic function $F(z)$ and the basic $g(t)$ are determined by the total strength of the spectral function,

$$\sigma(\omega)=\sum_i \sigma_i(\omega). \tag{2.31}$$

We now consider the field theory model of unstable particles, denoting the parent particle by c and two-body daughter particles by $b_1 b_2$. The decay interaction of the unstable particle of a momentum $\vec{q}$ is given by the Hamiltonian;

$$H_{\rm int}=\int \frac{d^3k_1 d^3k_2}{(2\pi)^6}(2\pi)^3\delta^3(\vec{q}-\vec{k}_1-\vec{k}_2) \times\frac{g}{\sqrt{8\omega_1\omega_2\omega_q}}[b_1^\dagger(\vec{k}_1)b_2^\dagger(\vec{k}_2)c(\vec{q})+(\text{H.c.})], \tag{2.32}$$

with g some coupling constant. We may then identify the decay product operator and the spectral function as

$$\sum_i \sqrt{\sigma_i(\omega,\vec{q})}b_i^\dagger(\omega,\vec{q})=\int \frac{d^3k_1d^3k_2}{(2\pi)^6}(2\pi)^3 \times\delta^3(\vec{q}-\vec{k}_1-\vec{k}_2)\,\delta(\omega-\omega_1-\omega_2)\frac{g}{\sqrt{8\omega_1\omega_2\omega_q}}b_1^\dagger(\vec{k}_1)b_2^\dagger(\vec{k}_2), \tag{2.33}$$

$$\sigma(\omega,\vec{q})=2\pi\sum_i \sigma_i(\omega,\vec{q})=\int \frac{d^3k_1d^3k_2}{(2\pi)^3}\delta^3(\vec{q}-\vec{k}_1-\vec{k}_2) \times\delta(\omega-\omega_1-\omega_2)\frac{g^2\prod_i (2\omega_i)^{F_i}}{8\omega_1\omega_2\omega_q}\sum_{\rm spins}|\mathcal{M}|^2, \tag{2.34}$$

where $F_i=1$ for fermions and $F_i=0$ for bosons. The decay amplitude $\mathcal{M}$ should be given separately in specific decay models. This can be thought of an extension of the discretely many (and finite) decay channel problem to the continuously many (and infinite) channel problem.

The field theory model of unstable particle decay is not exactly solvable, because the commutator among the decay product operator,

$$[b_1b_2,\;\; b_1^\dagger b_2^\dagger]=1\pm(b_1^\dagger b_1+b_2^\dagger b_2) \tag{2.35}$$

($\pm$ referring to the boson or fermion pair), is not the canonical one $[b,b^\dagger]=1$ as in the quantum mechanical model. Thus, the composite operators $b_i^\dagger(\omega,\vec{q})$ introduced by Eq. (2.33) do not obey the canonical commutation relation so crucial to the integrability. But the important case in which the bilinear term in the right hand side of the commutator can be neglected and the replacement is made,

$$[b_1b_2,\;\; b_1^\dagger b_2^\dagger]\to 1, \tag{2.36}$$

is identical to the solvable model in quantum mechanics. This occurs in the circumstance under which the thermal medium is very far away from the degeneracy limit. In this case the occupation number f_i which is the expectation value of the operators $b_i^\dagger b_i$ in the thermal medium is very small ($f_i \ll 1$), and one may neglect the bilinear term in the commutator above. Even in the dense, hot early universe of the standard cosmology the low occupation number is realized. We shall thus fully exploit this approximation in application to cosmology.

From the definition of the spectral function a relation to the decay rate follows:

$$\sigma(\omega_q,\vec{q})=\frac{M}{\omega_q}\frac{\Gamma}{2\pi}, \tag{2.37}$$

on the mass shell with $\omega=\omega_q=\sqrt{M^2+q^2}$. The factor M/ω_q represents the time dilatation effect. Off the mass shell,

$$\sigma(\omega,\vec{q}) = \frac{M}{\omega_q}\sigma(\sqrt{\omega^2-\vec{q}^2}), \tag{2.38}$$

where the spectral function in the right hand side $\sigma(\omega)$ is the one in the rest frame.

A choice of the decay model corresponds to a particular form of the spectral function. For instance, in the fermion-pair (of equal mass m) decay of a scalar boson $\varphi \to \psi\overline{\psi}$ described by a Lagrangian density of $\mathcal{L} = g\varphi\overline{\psi}\psi$,

$$\sigma(\omega,\vec{k}) = \frac{g^2}{16\pi^2\sqrt{k^2+M^2}}(\omega^2-k^2)\left(1-\frac{4m^2}{\omega^2-k^2}\right)^{3/2}. \tag{2.39}$$

A more general, convenient parametrization of the spectral function that becomes adequate in the temperature range of $T \gg 2m$ (the threshold for the decay product pair) is given, using the decay rate Γ;

$$\sigma(\omega,\vec{k}) = \frac{\Gamma}{2\pi}\frac{M}{\sqrt{k^2+M^2}}\frac{(\omega^2-k^2)^{\alpha/2}}{M^\alpha}. \tag{2.40}$$

For instance, the gauge X boson decay into a fermion-pair as well as the scalar X boson decay given by Eq. (2.39) has this form with $\alpha=2$, while the Higgs X boson decay into a boson-pair has this form with $\alpha=0$.

In reality, the spectral function in any non-trivial field theory model is complicated beyond the lowest order of perturbation. But it turns out that what is important for our subsequent analysis is the on-shell value of the spectral function given by the decay rate Γ and its behavior near the decay threshold, hence the parameter α in addition to the decay rate. The value of α is dictated by the unitarity relation for the opening channel, thus is essentially of kinematical origin. Both intermediate and late time behaviors of the decay, and important temperature dependent off-shell effects are described by these two parameters. Hence in a sense the detailed specification of a field theory for the decay is unnecessary.

The occupation number of the unstable particle decaying in vacuum is thus given by $|g(\vec{q},t)|^2$ with

$$g(\vec{q},t) = \frac{M}{\sqrt{M^2+q^2}}\int_{m_1+m_2}^\infty d\omega\ \sigma(\sqrt{\omega^2-q^2}) \times |F(\omega+i0^+,\vec{q})|^2 e^{i\omega t}, \tag{2.41}$$

where m_i are masses of decay product particles.

We should mention a limitation of our approach. We neglected in our model Hamiltonian the interaction of unstable particles with the medium, except the decay interaction. This guarantees a coherence of the decay and its inverse interaction and makes it easy to treat the environment effect on the decay process. There may however be an important class of thermal interactions on the unstable particle; the scattering off thermal light particles. If the thermal interaction of this sort is included, it gives rise to an additional term to the spectral function. In terms of the coupling strength this is a higher order effect of order $\alpha^2 T$ compared to the decay rate of order αM. However, this might contribute in the off-shell region. We shall go back to this effect when we discuss the off-shell effect in thermal medium in Sec. IV.

In the field theory of unstable particle decay one needs to perform renormalization. The method of renormalization is explained elsewhere [5] and here we shall write for simplicity quantities without renormalization, since in lowest order of perturbation renormalization is straightforward.

III. GENERALIZED BOLTZMANN EQUATION

In order to discuss the decay process of unstable particles that occur in the cosmic thermal medium, one must take into account the presence of medium and incorporate the inverse process that creates the unstable particle. A thermal environment is described by the density matrix denoted here by ρ_i and one has to consider how the decay proceeds in this mixed state. For the time being we assume that the change of the thermal environment is minor and the back reaction of the environment change against the decay and its inverse process is negligible. Later when we extend our analysis to baryogenesis, we incorporate a relevant effect of the environment change.

In the mixed state the occupation number of an excited level is

$$f(t) = \mathrm{tr}(c^\dagger(t)c(t)\rho_i). \tag{3.1}$$

It is not difficult to show from the operator solution that this quantity obeys the first order differential equation;

$$\begin{aligned}\frac{df}{dt} - 2\Re\frac{\dot{g}}{g}f = {} & i\int_{\omega_c}^\infty d\omega\sqrt{\sigma(\omega)}g^*\left(g + i\omega h e^{i\omega t} - \frac{\dot{g}}{g}he^{i\omega t}\right) \\ & \times\langle b^\dagger(\omega)c\rangle_i + (\mathrm{c.c}) + \int_{\omega_c}^\infty d\omega\int_{\omega_c}^\infty d\omega'\sqrt{\sigma(\omega)\sigma(\omega')} \\ & \times\Big(gh^*(\omega',t)e^{-i\omega t} + g^*h(\omega,t)e^{i\omega' t} \\ & + i(\omega-\omega')h(\omega,t)h^*(\omega',t) - 2\Re\frac{\dot{g}}{g}\ h(\omega,t)h^*(\omega',t)\Big) \\ & \times e^{i(\omega-\omega')t}\langle b^\dagger(\omega)b(\omega')\rangle_i .\end{aligned} \tag{3.2}$$

We used the notation

$$\langle A\rangle_i \equiv \mathrm{tr}(A\rho_i). \tag{3.3}$$

Although one can write $f(t)$ in an integrated form, this differential equation is more useful when one incorporates effect of the cosmic expansion. Another advantage of this form is that the initial state dependence via $\langle c^\dagger c\rangle_i$ is eliminated in favor of the occupation number $f(t)$ at any time t. It is however convenient not to eliminate the initial state dependence of environment variables, $\langle b^\dagger(\omega)b(\omega')\rangle_i$, when we later incorporate the environment change.

We take the uncorrelated initial state satisfying

$$\langle c^\dagger c\rangle_i = f(0), \quad \langle c^\dagger b(\omega)\rangle_i = 0,$$

$$\langle b^\dagger(\omega)b(\omega')\rangle_i = f_i(\omega)\delta(\omega-\omega'). \tag{3.4}$$

Although the unstable particle may or may not be in thermal equilibrium at much higher temperatures in the earlier epoch, the choice of the uncorrelated initial state seems reasonable, because there exists a time lag between the unstable and the decay product particles due to different interaction among themselves and with the rest of the environment (not written in the Hamiltonian above). We imagine that when unstable particles are about to decay, the decay interaction cannot keep pace with fast thermal interaction in the bulk of medium, hence we assume that the unstable particle has no thermal contact with the medium when we start calculation of the abundance evolution. In the end of the next section we however estimate how the scattering off thermal particles may affect the decay law in thermal medium.

Introducing the rate defined by

$$\Gamma(t) \equiv -2\Re\frac{\dot{g}}{g} = -\frac{d}{dt}\ln|g(t)|^2 \tag{3.5}$$

(which reduces to a constant rate Γ in the pole dominance approximation), one has

$$\frac{df}{dt} + \Gamma(t)f = \int_{\omega_c}^{\infty} d\omega\ \sigma(\omega)[2\Re(g(t)h^*(\omega,t)e^{-i\omega t}) + \Gamma(t)|h(\omega,t)|^2]\ f_i(\omega). \tag{3.6}$$

It is instructive to first give the time evolution in the narrow width approximation and then explain an improved approximation incorporating the C_1 integral. Using

$$g_0(t) \approx e^{-\Gamma t/2 + iE_1 t}, \tag{3.7}$$

one has

$$2\Re(g_0 h_0^* e^{-i\omega t}) - 2\Re\frac{\dot{g}_0}{g_0}|h_0|^2 \approx \frac{1}{(\omega-E_1)^2 + \frac{\Gamma^2}{4}}\ [\Gamma(1-e^{-\Gamma t/2}\cos(\omega-E_1)t) + 2e^{-\Gamma t/2}(\omega-E_1)\sin(\omega-E_1)t]. \tag{3.8}$$

There are oscillatory terms of frequency of $\approx 1/(\omega-E_1)$, but they are averaged out by the ω integration. From Eq. (3.6) the fundamental equation in thermal medium is found to be

$$\frac{df}{dt} = -\Gamma(f - f_i(E_1)), \tag{3.9}$$

in the narrow width approximation. Thus, this equation describes the relaxation towards the environment value by the constant rate Γ.

The evolution equation (3.6) correctly describes the relaxation process of unstable particle in medium beyond the narrow width approximation. Since $g(t)\to 0$ and $h(\omega,t)$ approaches an equilibrium value as $t\to\infty$, the stationary value of the occupation number is

$$f(t) \to \int_{\omega_c}^{\infty} d\omega\ \sigma(\omega)|F(\omega+i0^+)|^2 f_i(\omega) \equiv f_\infty. \tag{3.10}$$

If one takes a thermal distribution for the initial density matrix,

$$f_\infty = \int_{\omega_c}^{\infty} d\omega \frac{\sigma(\omega)}{(\omega-E_1+\Pi(\omega))^2 + (\pi\sigma(\omega))^2}\frac{1}{e^{\beta\omega}-1}. \tag{3.11}$$

The narrow width approximation $\sigma(\omega) \ll E_1$ of this gives the well-known result

$$f_\infty \approx \frac{1}{e^{\beta E_1}-1}, \tag{3.12}$$

but this form is only approximate and is not a good one in low temperatures. In low temperatures of $T \ll E_1$, the ω integral for f_∞ is dominated by the contribution near the threshold; taking the form of the threshold rise $\sigma(\omega) \approx c\ (\omega - \omega_c)^\alpha$ gives

$$f_\infty \approx \frac{c}{(E_1-\omega_c)^2}\int_{\omega_c}^{\infty} d\omega \frac{(\omega-\omega_c)^\alpha}{e^{\beta\omega}-1}. \tag{3.13}$$

When $T \ll \omega_c$ or $T \gg \omega_c$, this further simplifies to

$$f_\infty \approx \frac{c\ \Gamma(\alpha+1)}{(E_1-\omega_c)^2} e^{-\beta\omega_c} T^{\alpha+1}, \quad \text{for } T \ll \omega_c, \tag{3.14}$$

where $\Gamma(x)$ is the Euler's gamma function, and

$$f_\infty \approx \frac{c\ \zeta(\alpha+1)\Gamma(\alpha+1)}{(E_1-\omega_c)^2} T^{\alpha+1}, \quad \text{for } E_1 \gg T \gg \omega_c, \tag{3.15}$$

with $\zeta(x)$ the Riemann's zeta function. The Boltzmann suppressed temperature dependence of the occupation number $e^{-E_1/T}$ near $T = E_1$ is thus changed to the power behaved $T^{\alpha+1}$ at low temperatures [2]. What caused this big change is that the full Breit-Wigner shape is cut off effectively at the temperature T, since $T \ll$ the center location of the Breit-Wigner function.

We now turn to the unstable particle decay. The occupation number for a mode $\vec{k}$ is

$$f(\vec{k},t)=|g(\vec{k},t)|^2 f_{\rm th}(\omega_k)+\int d\omega |h(\omega,\vec{k},t)|^2 \times \int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\frac{g^2\prod_i (2\omega_i)^{F_i}}{8k_1k_2\omega_k} \times \sum_{\rm spins}|\mathcal{M}|^2\, f_{\rm th}(k_1)f_{\rm th}(k_2)\,\delta^3(\vec{k}-\vec{k}_1-\vec{k}_2) \times \delta(\omega-k_1-k_2). \tag{3.16}$$

We took the massless particle for decay products such that $\omega_i=|\vec{k}_i|$. The stationary limit of the occupation number is then

$$f_\infty(\vec{k})=\int d\omega |h(\omega,\vec{k},\infty)|^2 \int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\frac{g^2\prod_i (2\omega_i)^{F_i}}{8k_1k_2\omega_k} \times \sum_{\rm spins}|\mathcal{M}|^2\, f_{\rm th}(k_1)f_{\rm th}(k_2)\,\delta^3(\vec{k}-\vec{k}_1-\vec{k}_2) \times \delta(\omega-k_1-k_2). \tag{3.17}$$

In the low occupation number limit this is further simplified since

$$f_{\rm th}(k_1)f_{\rm th}(k_2)=e^{-\beta(k_1+k_2)},$$

which is replaced by $e^{-\beta\omega}$ in the above integrand. It is thus found that

$$f_\infty(\vec{k})\approx\int_k^\infty d\omega\frac{\sigma(\omega,\vec{k})e^{-\beta\omega}}{(\omega-\omega_k)^2+\pi^2\sigma^2(\omega,\vec{k})}. \tag{3.18}$$

Separating the pole and the threshold regions for this ω integral, one has

$$f_\infty(\vec{k})\approx e^{-\beta\omega_k}+\frac{\Gamma\, k^{\alpha+1}}{2\pi\, M^{\alpha+2}}\int_1^\infty dx(x^2-1)^{\alpha/2}e^{-\beta k\, x}. \tag{3.19}$$

The last integral in this equation is given by the modified Bessel function, and in the large k/T limit it behaves as

$$k^{\alpha/2}e^{-k/T}. \tag{3.20}$$

This has a narrower spread of the momentum than of order $\sqrt{MT}$ for the Maxwell-Boltzmann distribution, and its average is of order T [actually $(\alpha/2+3)\times T$ using the distribution, (3.20)]. We shall later use this fact to simplify the momentum dependence of the effective decay rate at late times. We plot the momentum distribution given by Eq. (3.19) in Fig. 2 to compare with the on-shell distribution relevant at low temperatures; $e^{-\sqrt{k^2+M^2}/T}$.

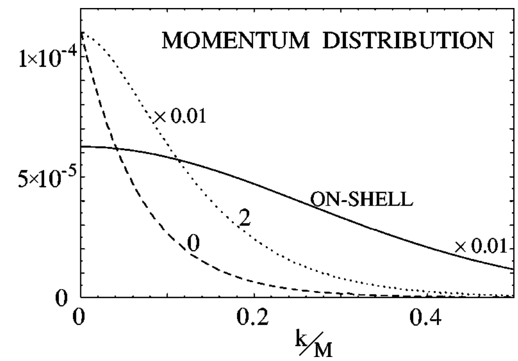

FIG. 2. Momentum distribution for a temperature and a decay rate, $T=0.07\ M$, $\Gamma=0.01\ M$. A comparison is made between the on-shell distribution function, $e^{-\sqrt{k^2+M^2}/T}$ appropriate at low temperatures, and the off-shell distribution of $\alpha=0$ and $\alpha=2$. Two of these distributions should be multiplied by 0.01 to get the correct values.

The total number density $n(t)$ is then obtained by summing the occupation number over modes of various particle momentum $\vec{k}$. In an isotropic medium it is

$$n(t)=\frac{1}{2\pi^2}\int_0^\infty dk\ k^2 f(k,t). \tag{3.21}$$

In general, various quantities in Eq. (3.6) depends on $\vec{k}$. This dependence is traced to the boost of the parent unstable particle. We shall often suppress the $\vec{k}$ dependence, unless otherwise there is a confusion. The stationary number density n_∞ is determined using f_∞ for $f(\vec{k},t)$; in the two-particle decay of equal mass m,

$$n_\infty=\frac{1}{2\pi^2}\int_0^\infty dk\ k^2\int_{\sqrt{k^2+4m^2}}^\infty d\omega \times\frac{\sigma(\omega,k)}{(\omega-\sqrt{k^2+M^2})^2+\pi^2\sigma^2(\omega,k)}\frac{1}{e^{\omega/T}-1}. \tag{3.22}$$

Throughout this work we ignored a small part of $\Pi(\omega)$; the residual $\Pi(\omega)$ that remains after renormalization.

In the low temperature range of $M\gg T\gg 2m$ the time dilatation effect is negligible and

$$n_\infty\approx\frac{\Gamma}{4\pi^3}\int_0^\infty d\omega\frac{1}{e^{\beta\omega}-1}\int_0^\omega dk\ k^2\frac{(\omega^2-k^2)^{\alpha/2}}{M^{\alpha+2}}. \tag{3.23}$$

It is then analytically calculated as

$$\frac{n_\infty}{T^3}\approx A(\alpha)\frac{\Gamma}{M}\left(\frac{T}{M}\right)^{\alpha+1},$$

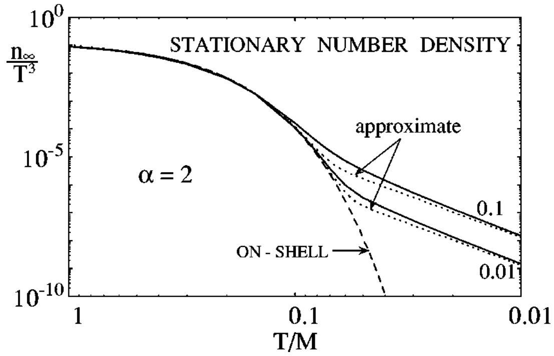

FIG. 3. Stationary number density given by Eq. (3.22) which is divided by temperature^3 for two values of the decay rate, $\Gamma/M=0.1,0.01$. The dotted lines are calculated using the approximate formula, Eq. (3.25), while the broken line is the on-shell contribution alone, the first term in this equation.

$$A(\alpha)=\frac{\zeta(\alpha+4)\Gamma(\alpha+4)\Gamma\left(\frac{\alpha}{2}+1\right)}{16\pi^2\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}+\frac{5}{2}\right)}. \tag{3.24}$$

In Fig. 3 we plotted the stationary number density, Eq. (3.22) combined with the spectral function (2.40), as a function of T/M for two values of the decay rate Γ. Our numerical computation supports that to a good accuracy the stationary number density n_∞ is given by a sum of the pole and the threshold contribution;

$$n_\infty=\frac{1}{2\pi^2}\int_0^\infty dk\frac{k^2}{e^{\sqrt{k^2+M^2}/T}-1}+A(\alpha)\frac{\Gamma}{M}\left(\frac{T}{M}\right)^{\alpha+1}T^3. \tag{3.25}$$

This formula is accurate for any temperature T less than M if the decay rate Γ/M is small enough. But at higher temperatures the first on-shell term in Eq. (3.25) alone is accurate and the off-shell power term $\propto T^{\alpha+4}$ in this equation should be discarded.

One may define the equal temperature T_{eq} as the one at which the two contributions in this equation, the pole and the threshold contributions, become equal. How this equal temperature depends on the rate Γ/M is important, and it is shown in Fig. 4 for two values of α. With the on-shell contribution given by

$$n_\infty\approx\left(\frac{MT}{2\pi}\right)^{3/2}e^{-M/T}, \tag{3.26}$$

the equation that determines $T_{\rm eq}$ is

$$x^{\alpha+5/2}e^{-x}=(2\pi)^{3/2}A(\alpha)\frac{\Gamma}{M},\quad x\equiv\frac{M}{T_{\rm eq}}. \tag{3.27}$$

for a small Γ/M. A rough analytic estimate in the $\Gamma\to 0$ limit would be

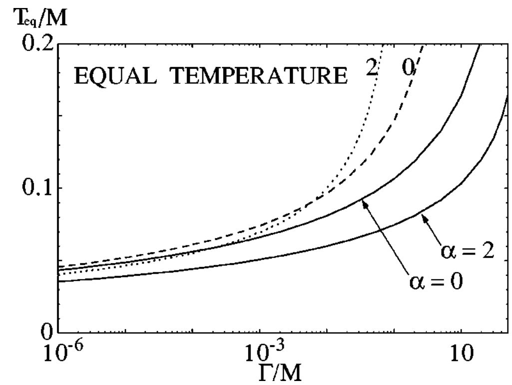

FIG. 4. The equal-temperature at which the on-shell and the off-shell contributions become equal is shown for two values of $\alpha=0,2$. The dashed and the dotted lines are results of the approximate formula, Eq. (3.28).

$$\frac{T_{\rm eq}}{M}\approx\left(\ln\frac{M}{(2\pi)^{3/2}A(\alpha)\Gamma}+\left(\alpha+\frac{5}{2}\right)\ln\ln\frac{M}{(2\pi)^{3/2}A(\alpha)\Gamma}\right)^{-1}, \tag{3.28}$$

but this expression is accurate only for a very small Γ/M, for instance, $\Gamma/M<10^{-4}$ for $\alpha=0$.

We now turn to the time evolution of the occupation number and the number density. The approach towards the stationary value is governed by the large time limit of $g(t)$. In the pole dominance approximation the relaxation is exponential in time and fast. On the other hand, the true late time behavior is the power law, and this makes complete relaxation slower [2].

The effect of the cosmological expansion is readily incorporated for the number density n by adding the $3H\ n$ term where the Hubble rate

$$H=\frac{\dot{a}}{a}=\frac{1}{2t}. \tag{3.29}$$

We first give the result in the narrow width approximation;

$$\frac{dn}{dt}+3\frac{\dot{a}}{a}n=-\overline{\Gamma}(n-n^{\rm th}(T)), \tag{3.30}$$

$$n^{\rm th}(T)=\frac{T^3}{2\pi^2}\int_0^\infty dx\frac{x^2}{e^{\sqrt{x^2+M^2/T^2}}-1}. \tag{3.31}$$

We introduced an averaged quantity over the momentum in order to simplify the mode integral;

$$\overline{\Gamma}=\frac{1}{2\pi^2 n(t)}\int_0^\infty dk\ k^2\ \Gamma_k\ f(k,t). \tag{3.32}$$

In low enough temperatures this averaging is indeed simple; $\overline{\Gamma}\approx\Gamma$, because the time dilatation effect given by $M/\sqrt{M^2+k^2}\approx 1-k^2/2M$ is negligible for the slow motion of parent particles. The equation for the number density should be combined with the well known temperature-time relation in the radiation dominated universe,

$$t=\frac{dm_{\rm pl}}{T^2},\quad \frac{\dot a}{a}=-\frac{\dot T}{T},\quad d=\sqrt{\frac{45}{16\pi^3\,N}},\tag{3.33}$$

with N the number of massless species contributing to the energy density.

In the low temperature region of $T<M$ the equation is further simplified by using the Maxwell-Boltzmann distribution function of the zero chemical potential,

$$e^{-(M+k^2/2M)/T}.\tag{3.34}$$

A convenient evolution equation is obtained using the dimensionless quantities

$$Y\equiv\frac{n}{T^3},\quad u\equiv\frac{M}{T}=\sqrt{\frac{M^2t}{dm_{\rm pl}}},\quad \tau\equiv\Gamma t.\tag{3.35}$$

The time scale of variation is given by the lifetime Γ^{-1}, hence it is useful to use the dimensionless time τ. The evolution equation is then

$$\frac{dY}{d\tau}=-(Y-S),\quad S(u)=\left(\frac{u}{2\pi}\right)^{3/2}e^{-u}.\tag{3.36}$$

The source term varies as $S(\sqrt{\tau/\eta})$, with

$$\eta=\frac{dm_{\rm pl}\Gamma}{M^2},\quad u=\sqrt{\frac{\Gamma t}{\eta}}=\sqrt{\frac{\tau}{\eta}}.\tag{3.37}$$

Thus, S as a function of τ has a maximum around $\tau=\eta$. The meaning of the quantity η is the decay rate Γ divided by the Hubble rate H at the temperature $T=M$.

One can readily integrate this equation. The solution to the differential equation for the yield Y is

$$Y(\tau)=\int_{\tau_i}^{\tau}d\tau'\;S\left(\sqrt{\frac{\tau'}{\eta}}\right)e^{-(\tau-\tau')}+Y(\tau_i)e^{-(\tau-\tau_i)}.\tag{3.38}$$

The first term gives the yield created from the thermal medium. For $\tau\gg1$, namely at times much larger than the lifetime, $t\gg1/\Gamma$, the dominant region of the τ' integral is $\tau-\tau'<1$ and the integral gives

$$Y(\tau)\approx S\left(\sqrt{\frac{\tau}{\eta}}\right).\tag{3.39}$$

This is in general valid unless $\eta<\eta_{\rm cr}$, where the critical value $\eta_{\rm cr}=\frac{1}{12}$. The yield Y thus roughly follows the thermal value $S(T)$.

There is some lesson one can learn on the more general case of the mode independent $\Gamma(t)=-(d/dt)\ln|g|^2$, from this calculation in the pole dominance approximation. It is obvious that even for the more general case the general solution to the mode-summed form of Eq. (3.6) extended to the expanding universe is given by

$$Y(t)=|g(t)|^2\left(\int_{t_i}^{t}dt'\frac{\Gamma(t')S(t')}{|g(t')|^2}+Y(t_i)\frac{1}{|g(t_i)|^2}\right),\tag{3.40}$$

where the source term $S(t)$ is given by

$$\frac{1}{2\pi^2T^3\Gamma(t)}\int_0^\infty dk\;k^2\int_{\omega_c}^\infty d\omega[2\Re(g(t)h^*(\omega,\vec k,t)e^{-i\omega t})+\Gamma(t)|h(\omega,\vec k,t)|^2]\;\frac{\sigma(\omega,\vec k)}{e^{\beta\omega}-1}.\tag{3.41}$$

The stationary value of this source term $S(t)$ at $t\to\infty$ is

$$S_\infty=\frac{n_\infty}{T^3},\tag{3.42}$$

where n_∞ is given by Eq. (3.22). Actually, this quantity is not stationary, since the temperature T gradually changes with the cosmological expansion.

When the rate Γ_k does depend on the momentum k, the solution for Y cannot be given in a simple closed form such as Eq. (3.40).

IV. OFF-SHELL EFFECT

The off-shell effect appears in two ways; first, in the slower time dependence of relaxation ($|g|\propto t^{-\alpha-1}$), and second, in a larger source term ($S\propto T^{\alpha+1}$ at small temperature T). Both are related to the threshold behavior of the spectral function, $\sigma\propto(\omega-\omega_c)^\alpha$.

We found elsewhere [5] that adding a power term $g_1(t)$ to the pole term $g_0(t)$ is an excellent approximation for $g(t)$ in the entire time range, unless one looks into the very short-time region of relaxation. The power law period is represented by

$$g_1(t)=\frac{i\;c\Gamma(\alpha+1)}{Q^2t^{\alpha+1}}e^{i(\omega_ct+\pi\alpha/2)},$$

$$k_1(\omega,t)=-i\int_t^\infty dt'g_1(t')e^{-i\omega t'},\tag{4.1}$$

where $\sigma(\omega)\approx c(\omega-\omega_c)^\alpha$ near the threshold and $Q=M-\omega_c$. With this power behavior, the rate $\Gamma(t)=-(d/dt)\ln|g(t)|^2$ is $\approx2(\alpha+1)/t$. The final yield $Y=n/T^3$ is then of order, $S_\infty\propto(\Gamma/M)(T/M)^{\alpha+1}$. Thus, the yield does not decrease as rapidly as might have been expected from the exponential decay law, but it decreases with a power, $T^{\alpha+1}\propto t^{-(\alpha+1)/2}$.

A complication arises when one incorporates dependence on the particle momentum. This effect appears in two ways; first, in the time dilatation of the lifetime ω_k/M and second, in the function $g(k,t)$. The time dilatation effect is negligible if one only considers the temperature range of $T<M$ (the mass of unstable particle). We shall thus discuss the momentum dependence of the off-shell contribution to g; $g_1(k,t)$. In

the C_1 contour integral of Fig. 1 one has an approximate expression of the form, with $\omega_c=k$,

$$g_1(k,t)\approx i\frac{e^{ikt}}{M^2}\int_0^\infty dy\ \sigma(k+iy,k)e^{-yt}. \tag{4.2}$$

We then use the spectral form (2.40) or its low temperature approximation,

$$\sigma(\omega,k)\approx\frac{\Gamma}{2\pi}\frac{(\omega^2-k^2)^{\alpha/2}}{M^\alpha}, \tag{4.3}$$

to get

$$g_1(k,t)\approx i(2i)^{\alpha/2}\Gamma\left(\frac{\alpha}{2}+1\right)\frac{\Gamma}{2\pi}\frac{k^{\alpha/2}e^{ikt}}{M^{\alpha+2}t^{\alpha/2+1}}. \tag{4.4}$$

An approximate evolution equation is summarized using the dimensionless time variable, $\tau\equiv\Gamma t$, and the time-temperature relation $T/M=\sqrt{\eta/\tau}$,

$$\frac{dY}{d\tau}=-\int_0^\infty\frac{dk\ k^2}{2\pi^2T^3}\gamma(k,t)\left(f_k(t)-\int_k^\infty d\omega\frac{\sigma(\omega,k)}{(\omega-\omega_k)^2+(\Gamma/2)^2}\frac{1}{e^{\beta\omega}-1}\right), \tag{4.5}$$

$$Y=\frac{1}{T^3}\int_0^\infty\frac{dk\ k^2}{2\pi^2}f_k(t), \tag{4.6}$$

$$\gamma(k,t)=-2\Re\frac{d}{\Gamma\ dt}\ln(g_0(t)+g_1(k,t))=-\frac{\frac{d}{\Gamma\ dt}|g_0(t)+g_1(k,t)|^2}{|g_0(t)+g_1(k,t)|^2}. \tag{4.7}$$

Thus, the momentum dependence $g_1(k,t)$ is convoluted with other momentum dependent functions in the $\vec{k}$ integral. In low temperatures the most important part of this momentum dependence is in the spectral function $\sigma(\omega,k)$ which vanishes at the threshold, $\omega_c=k$ for the massless daughter particles. In some sample numerical computations that include the phase factor in $g_1(k,t)$, we observe oscillatory behaviors around the transition time from the pole to the power period that occurs at $\approx T_{\rm eq}$ given by Eq. (3.28). In the rest of the time region the yield Y smoothly varies, and it is very well described by using a simplified effective rate,

$$\gamma(k,t)\rightarrow\overline{\gamma}(k,t)=-\frac{\frac{d}{\Gamma\ dt}(|g_0(t)|^2+|g_1(k,t)|^2)}{|g_0(t)|^2+|g_1(k,t)|^2}. \tag{4.8}$$

The oscillatory behavior in the transition region gets smoothed and approaches $\overline{\gamma}$ given above when one time averages the rate and increases the resolution time Δt towards the lifetime $1/\Gamma$. Moreover, the late time behavior is insensitive to whether one uses the exact rate $\gamma(k,t)$ or the effective rate $\overline{\gamma}(k,t)$ above. It is thus a reasonably good approximation to use the average rate (4.8).

In a still further simplification one may neglect the momentum dependence in $|g_1(k,t)|^2\propto k^\alpha$ and replace the momentum by its average in low temperatures; $k^a\rightarrow O[1]\times T^a$. The relation $\overline{k^\alpha}\propto T^\alpha$ is consistent with the late-time momentum distribution of the stationary occupation number $f_\infty(\vec{k})$, as discussed in Eq. (3.20). We have checked that the following crude rate equation,

$$\frac{dY}{d\tau}=-\gamma(Y-Y_0-S_0), \tag{4.9}$$

$$\gamma=\frac{e^{-\Gamma t}+(\alpha+2)B(\alpha)\left(\frac{\Gamma}{M}\right)^{\alpha+4}\left(\frac{T}{M}\right)^{\alpha}(\Gamma t)^{-\alpha-3}}{e^{-\Gamma t}+B(\alpha)\left(\frac{\Gamma}{M}\right)^{\alpha+4}\left(\frac{T}{M}\right)^{\alpha}(\Gamma t)^{-\alpha-2}}, \tag{4.10}$$

$$Y_0=\frac{1}{2\pi^2T^3}\int_0^\infty dk\frac{k^2}{e^{\sqrt{k^2+M^2}/T}-1}, \tag{4.11}$$

$$S_0=\frac{\zeta(\alpha+4)\Gamma(\alpha+4)\Gamma\left(\frac{\alpha}{2}+1\right)}{16\pi^2\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}+\frac{5}{2}\right)}\frac{\Gamma}{M}\left(\frac{T}{M}\right)^{\alpha+1}, \tag{4.12}$$

is a reasonably good approximation. We may use $B(\alpha)=\Gamma(\frac{3}{2}\alpha+3)/\Gamma(\frac{1}{2}\alpha+3)$, since $\overline{k^\alpha}=B(\alpha)T^\alpha$ for the late-time distribution function already discussed. The source term is separated into the on-shell contribution Y_0 plus the off-shell contribution S_0.

There is a second temperature T_* or time at which the off-shell effect becomes conspicuous. This is the time when the two terms of $g(t)$ becomes equal; $|g_0(t)|^2=\overline{|g_1(k,t)|^2}$. Using the formula above, one gets in the $\Gamma\rightarrow 0$ limit

$$\frac{T_*}{M}\approx\sqrt{\frac{\eta}{(\alpha+4)\ln\frac{M}{\Gamma}}}. \tag{4.13}$$

Unless η is small [roughly $\eta<1/\ln(M/\Gamma)$], $T_*>T_{\rm eq}$ usually. The transitional period then lasts for a while at $T_*>T>T_{\rm eq}$. Thus, T_* is the temperature at which the slower decrease of the remnant becomes apparent, while $T_{\rm eq}$ is the one at which the larger source term becomes visible.

The late time limit of solution to this equation can be analytically worked out. Since the on-shell source term Y_0 is exponentially small at late times when $T<T_{\rm eq}$,

$$\frac{dY}{d\tau}\approx-\frac{\alpha+2}{\tau}(Y-S_0). \tag{4.14}$$

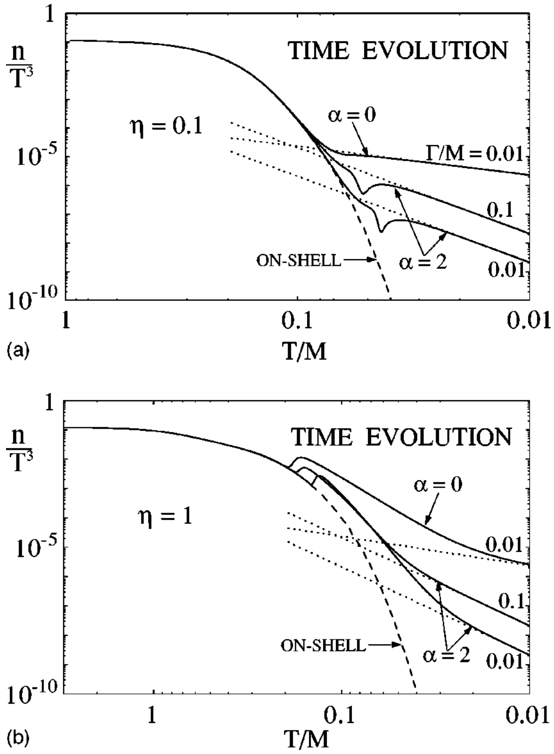

FIG. 5. Time evolution of the yield $Y=n/T^3$ for different values of η; 0.1 (a) and 1 (b) and for different decay rates, $\Gamma/M=0.1,0.01$. For comparison the on-shell evolution (the broken line) and the late time form, Eq. (4.15) (the dotted line) are shown.

This equation is readily solvable, and in the $\tau\to\infty$ limit

$$Y\to\frac{2\alpha+4}{\alpha+3}\frac{n_\infty}{T^3}=\frac{2\alpha+4}{\alpha+3}A(\alpha)\frac{\Gamma}{M}\left(\frac{T}{M}\right)^{\alpha+1}. \tag{4.15}$$

This asymptotic form becomes relevant, starting at $T=T_{\rm eq}$.

We show some numerical results in Fig. 5 and compare with the simple analysis. Besides α, important parameters in the time evolution are the rate Γ/M and η that appears in the time-temperature relation (3.37). It is clearly seen that the yield $Y=n/T^3$ is accurately given by the on-shell formula at high temperatures and by the power law formula (4.15) at low temperatures, with a transition region at $T/M\approx(\text{several}\times10)^{-1}$.

One may summarize this result, by saying that the time evolution gives the yield $Y(t)$, starting from the stationary value $S_\infty(T)$ at high temperatures and ending at $(2\alpha+4)/(\alpha+3)S_\infty(T)$ at low temperatures. The transient temperature is characterized by the temperature of order $T_{\rm eq}-T_*$ and the yield below this temperature follows

$$Y\approx Y_{\rm eq}\left(\frac{T}{T_{\rm eq}}\right)^{\alpha+1}. \tag{4.16}$$

Importance of the off-shell effect is measured by how close the temperature Max$(T_*,T_{\rm eq})$ for the onset of the off-shell effect is to the temperarure scale for the decay. This temperature is estimated as follows. One first defines T_d by $\Gamma=H(T_d)$, which gives $T_d=\sqrt{\eta}M$. We then discuss two cases separately. First, if $\eta\gg1$, $T_d\gg M$ and the decay and its inverse decay frequently occur at temperatures between T_d and the threshold M, maintaining the thermal abundance $Y=Y_0$. At $T\ll M$ the inverse decay is less frequent than the decay and there is a Boltzmann suppression $e^{-M/T}$ until the temperature decreases down to T_*. (In the case of $\eta\gg1$, $T_*>T_{\rm eq}$ always.) Thus, the importance of the off-shell effect is measured by how large is the quantity,

$$\frac{T_*}{M}\approx\sqrt{\frac{\eta}{\ln(M/\Gamma)}}, \tag{4.17}$$

if this number is less than unity. It however often happens that $\eta\gg1$ is huge and $T_*\gg M\gg T_{\rm eq}$, as it occurs for instance in the top and the weak boson decay. In this case a true measure of the off-shell importance is given by how large the value of $T_{\rm eq}/M\approx1/(\ln M/\Gamma)$ is.

On the other hand, if $\eta\ll1$, then $T_d\ll M$, and the decay does not occur until $T\ll T_d$, much below the Q value. In this case the off-shell importance is determined by how large

$$\frac{\text{Max}(T_{\rm eq},T_*)}{T_d}=\text{Max}\left(\frac{1}{\sqrt{\eta}\ln\frac{M}{\Gamma}},\frac{1}{\sqrt{\ln\frac{M}{\Gamma}}}\right) \tag{4.18}$$

is.

In these two cases of both large and small η, a large decay constant of for example, $\Gamma/M=1-10^{-2}$, is expected to give a large off-shell effect irrespective of the η value. This is physically reasonable since for a large coupling the narrow width approximation is expected to break down.

We now turn to the scattering effect mentioned in Sec. II. We first give a crude estimate and later elaborate more quantitatively. The loss of coherence due to the scattering off thermal particles occurs in an inverse time scale of order

$$\sigma_s\approx n_{\rm th}(T)\frac{\overline{v\Sigma}}{2\pi}, \tag{4.19}$$

where $\overline{v\Sigma}$ is the averaged cross section of the scattering interaction with the thermal medium. This gives a new term to the spectral σ of order

$$\sigma_s\approx\frac{4(2s+1)\zeta(3)}{3\pi^2}\frac{\alpha_s^2T^3}{M^2}, \tag{4.20}$$

where α_s is the strongest dimensionless coupling with the medium and $2s+1$ is the spin degrees of freedom. We took the Thomson type cross section $\propto\alpha_s^2/M^2$. This is a small addition to the on-shell rate $\Gamma/2\pi$, due to the extra coupling factor and the temperature suppression. More important is its possible contribution to the late-time and the low-temperature region. Its contribution to the occupation number in the stationary limit,

$$\delta f_\infty\approx\frac{1}{M^2}\int_{\omega_c}^\infty d\omega\ \sigma_s e^{-\beta\omega}, \tag{4.21}$$

gives

$$\delta f_\infty \approx \frac{4(2s+1)\zeta(3)}{3\pi^2}\alpha_s^2\left(\frac{T}{M}\right)^4. \tag{4.22}$$

We have so far ignored the momentum dependence of the spectral function $\sigma_s(\omega,\vec{q})$. Indeed, the above estimate of the spectral function is valid only near the mass shell. The correct formula including the off-shell contribution is

$$\sigma_s(\omega,\vec{q}) = \frac{v\Sigma_s}{2\pi}\int \frac{d^3k}{(2\pi)^3}e^{-\beta k}\theta(\omega+k-\sqrt{(\vec{k}+\vec{q})^2+M^2}). \tag{4.23}$$

This gives to the stationary number density an extra off-shell term for the scattering effect,

$$\delta n_\infty = \frac{1}{M^2}\int \frac{d^3q}{(2\pi)^3}\int d\omega\ e^{-\beta\omega}\ \sigma_s(\omega,\vec{q}) \approx \frac{2}{3}v\Sigma_s MT\ n_{\rm th}(T). \tag{4.24}$$

This formula is valid only when deviation from the thermal distribution is small. Compared to the on-shell value $n_{\rm th}$, this δn_∞ is smaller by a factor $\alpha_s^2(T/M)$. Thus, when the off-shell contribution of order, $(\Gamma/M)(T/M)^{\alpha+1}T^3$, dominates over the on-shell value $n_{\rm th}$, the scattering effect to the source term S may be ignored.

The scattering effect also gives a new late-time contribution;

$$\int \frac{d^3q}{(2\pi)^3}\delta g(\vec{q},t) = \frac{1}{M^2}\int \frac{d^3q}{(2\pi)^3}\int d\omega\ \sigma_s(\omega,\vec{q})e^{i\omega t} \approx \frac{2(2s+1)}{3\pi^2}\alpha_s^2\frac{T}{M^2\ t}n_{\rm th}, \tag{4.25}$$

to be compared to the previous off-shell contribution,

$$\int \frac{d^3q}{(2\pi)^3}|g_1(\vec{q},t)| \approx \frac{\Gamma}{2\pi}\left(\frac{T}{M}\right)^{\alpha/4+3/2}\frac{M^2}{(Mt)^{\alpha/2+1}}. \tag{4.26}$$

Note that $n_{\rm th}(T) < (\Gamma/M)(T/M)^{\alpha+1}T^3$ in the temperature region of our interest. Thus, the scattering effect appears negligible, because the temperature inequality usually obeyed,

$$\frac{T}{M} < O\left[\left(\frac{\Gamma}{\eta\alpha_s^2\ M}\right)^{1/3}\right], \tag{4.27}$$

gives a smaller late-time contribution than Eq. (4.26).

V. SPECIES EVOLUTION EQUATION AND BARYOGENESIS

It is of great interest if one can predict consequences of the modified abundance of unstable particles during their decay that become permanently imprinted in the rest of the cosmic evolution. In this respect it is important to clarify a possible change of the thermal environment due to the decay. The mere increase of the total number of decay products is hardly recognizable in thermal medium. One should examine a more detailed distribution of particle species in the process of unstable particle decay. As an important example of this class, we shall discuss baryogenesis in a simplified model.

To this end we need to introduce several decay modes distinguished by a flavor index j of the continuous state $b_j^\dagger(\omega)|0\rangle$. In the problem of baryogenesis we think of many channels of different baryon numbers such as the diquark (qq) and the quark-lepton pair $(\overline{ql})$, considering the decay of anti-X boson along with the X-boson. It is presumably better to consider channels of different $B-L$ such as $N \to l\overline{H}$, $\overline{l}H$ (where N is a heavy right-handed Majorana neutrino, and H is the Higgs doublet), in view of that the baryon and the lepton numbers are redistributed at lower temperatures by the baryon nonconserving electroweak process keeping $B-L$ unchanged [6–9].

The basic assumption taken in estimating the environment change is that there exists baryon conserving strong interaction among environment particles such that the kinetic equilibrium among them is readily established, leading to the environment particle distribution described by a thermal distribution function of a finite chemical potential μ associated with the baryon number,

$$\langle b_j^\dagger(\omega)b_j(\omega)\rangle_i = \frac{1}{e^{\beta(\omega-\alpha_j\mu)}-1}. \tag{5.1}$$

Here α_j is the baryon number of the species j. In subsequent application to baryogenesis the limit of a small chemical potential is relevant. Thus,

$$\langle b_j^\dagger(\omega)b_j(\omega)\rangle_i \approx \left(1-\alpha_j\mu\frac{d}{d\omega}\right)\frac{1}{e^{\beta\omega}-1}. \tag{5.2}$$

It is useful to note the conservation of particle number,

$$\frac{d}{dt}\left(c^\dagger(t)c(t)+\int_{\omega_c}^\infty d\omega\sum_j\ b_j^\dagger(\omega,t)b_j(\omega,t)\right)=0, \tag{5.3}$$

which holds as an operator identity. It is easy to confirm that if

$$\langle b_j^\dagger(\omega)b_k(\omega')\rangle_i \propto \delta_{jk}\delta(\omega-\omega'),$$

$$\frac{d}{dt}\int_{\omega_c}^\infty d\omega\sum_j\ \langle b_j^\dagger(\omega,t)b_j(\omega,t)\rangle \to 0, \tag{5.4}$$

as $t\to\infty$, but a net baryon number may be generated, since

$$\frac{d}{dt}\int_{\omega_c}^{\infty}d\omega\sum_j \alpha_j\langle b_j^\dagger(\omega,t)b_j(\omega,t)\rangle\neq 0. \tag{5.5}$$

One can write down a set of time evolution equation of each species along with the one of the parent unstable particle. These equations are not particularly illuminating. We shall directly work out the situation relevant to baryogenesis. In this problem the environment is gradually changed by the presence of a small chemical potential associated with the baryon number. We imagine that the bulk of thermalization processes not included in the decay interaction conserves the baryon number, and the time variation of the net baryon number is driven by the asymmetry generated by the pair decay of X and $\overline{X}$. One thus expands in power series of the small chemical potential and identify the baryon number density of the thermal environment;

$$n_B=\frac{1}{2\pi^2}\int_0^{\infty}dk\ k^2\sum_j\left(\frac{\alpha_j}{e^{\beta(\omega_k-\alpha_j\mu)}-1}+(\alpha_j\to-\alpha_j)\right)$$
$$\approx-\frac{\mu}{\pi^2}\sum_j \alpha_j^2\int_0^{\infty}dk\ k^2\frac{d}{d\omega_k}\frac{1}{e^{\beta\omega_k}-1}. \tag{5.6}$$

Furthermore, we only consider for simplicity the case of many decay modes whose rates differ only in the overall partial rate, $\sigma_j(\omega)=\gamma_j\ \sigma(\omega)$, where the species independent $\sigma(\omega)$ is the total spectral function common to X and $\overline{X}$ due to the CPT theorem. Thus $\Sigma_j\gamma_j=\Sigma_j\overline{\gamma_j}=1$. This approximation is excellent if the decay products are much lighter than the unstable particle, as it happens in the X decay into ordinary fermions. The fundamental baryon asymmetry ϵ when a pair of X and $\overline{X}$ bosons decay arises via a combined effect of baryon nonconservation and CP violation [10] such that for some partial rates $\gamma_j\neq\overline{\gamma_j}$. It is given by

$$\epsilon\equiv\sum_j \alpha_j(\gamma_j-\overline{\gamma_j}). \tag{5.7}$$

We consider both the fermion-pair decay [11] and the boson-pair decay [12] of X boson. It is thus convenient to parametrize the spectral function as in Eq. (2.40),

$$\sigma(\omega,k)=\frac{\Gamma}{2\pi}\frac{M}{\omega_k}\frac{(\omega^2-k^2)^{\alpha/2}}{M^{\alpha}}, \tag{5.8}$$

where $\alpha=2$ for the fermion-pair decay and $\alpha=0$ for the boson-pair decay.

To lowest order of the asymmetry ϵ and the chemical potential μ, the basic evolution equation is

$$\left(\frac{d}{dt}+3\frac{\dot a}{a}\right)n_B=\int_0^{\infty}\frac{dk\ k^2}{2\pi^2}\Bigg(\Gamma_k(t)\left(\frac{\epsilon}{2}(f_X+f_{\overline{X}})+\frac{\delta}{2}(f_X-f_{\overline{X}})\right)-\epsilon\int_{\omega_c}^{\infty}d\omega\ \sigma(\omega)[\Gamma_k(t)|h(\omega,t)|^2$$
$$+2\Re(g(t)h^*(\omega,t)e^{-i\omega t})]f^{\rm th}(\omega)+\frac{\mu\delta^2}{2}\int_{\omega_c}^{\infty}d\omega\ \sigma(\omega)[\Gamma_k(t)|h(\omega,t)|^2+2\Re(g(t)h^*(\omega,t)e^{-i\omega t})]\frac{df^{\rm th}(\omega)}{d\omega}$$
$$+2\mu\left(\sum_j \alpha_j^2(\gamma_j+\overline{\gamma_j})-\frac{\delta^2}{2}\right)\int_{\omega_c}^{\infty}d\omega\ \sigma(\omega)\Re h(\omega,t)\frac{df^{\rm th}(\omega)}{d\omega}\Bigg), \tag{5.9}$$

$$\left(\frac{d}{dt}+3\frac{\dot a}{a}\right)(n_X+n_{\overline{X}})=\int_0^{\infty}\frac{dk\ k^2}{2\pi^2}\left(-\Gamma_k(t)(f_X+f_{\overline{X}})+2\int_{\omega_c}^{\infty}d\omega[\Gamma_k(t)|h(\omega,t)|^2+2\Re(g(t)h^*(\omega,t)e^{-i\omega t})]\sigma(\omega)f^{\rm th}(\omega)\right), \tag{5.10}$$

$$\left(\frac{d}{dt}+3\frac{\dot a}{a}\right)(n_X-n_{\overline{X}})=\int_0^{\infty}\frac{dk\ k^2}{2\pi^2}\left(-\Gamma_k(t)(f_X-f_{\overline{X}})-\mu\delta\int_{\omega_c}^{\infty}d\omega\sigma(\omega)[\Gamma_k(t)|h(\omega,t)|^2+2\Re(g(t)h^*(\omega,t)e^{-i\omega t})]\frac{df^{\rm th}(\omega)}{d\omega}\right). \tag{5.11}$$

Here

$$f^{\rm th}(\omega)=\frac{1}{e^{\beta\omega}-1} \tag{5.12}$$

is the thermal occupation number for the zero chemical potential and

$$\delta=\sum_j \alpha_j(\gamma_j+\overline{\gamma_j}). \tag{5.13}$$

We first consider the pole dominance approximation. In this approximation a simple relation,

$$\Re h_p=\Re\left(-\frac{\dot g}{g}|h|^2+gh^*e^{-i\omega t}\right)_p, \tag{5.14}$$

holds. Using the relation of the baryon number density and the chemical potential,

$$n_B=\frac{1}{3}\sum_j \alpha_j^2\mu T^2, \tag{5.15}$$

the relevant evolution equation is

$$\left(\frac{d}{dt}+3\frac{\dot{a}}{a}\right)n_B=\epsilon\,\Gamma(n_+-n_0^{\rm th})+\Gamma\,\delta\,n_- -\frac{K}{2}\Gamma\frac{n_0^{\rm th}}{T^3}n_B, \tag{5.16}$$

$$\left(\frac{d}{dt}+3\frac{\dot{a}}{a}\right)n_+=-\Gamma(n_+-n_0^{\rm th}), \tag{5.17}$$

$$\left(\frac{d}{dt}+3\frac{\dot{a}}{a}\right)n_-=-\Gamma n_- +\frac{\tilde{\delta}\Gamma}{2}\frac{n_0^{\rm th}}{T^3}n_B, \tag{5.18}$$

$$n_0^{\rm th}=\left(\frac{MT}{2\pi}\right)^{3/2}e^{-M/T}, \tag{5.19}$$

$$n_+=\int_0^\infty\frac{dk\,k^2}{4\pi^2}(f_X+f_{\bar{X}}),\quad n_-=\int_0^\infty\frac{dk\,k^2}{4\pi^2}(f_X-f_{\bar{X}}), \tag{5.20}$$

$$K=\frac{6\sum_j \alpha_j^2(\gamma_j+\overline{\gamma_j})}{\sum_j \alpha_j^2}>0,\quad \tilde{\delta}=\frac{3\delta}{\sum_j\alpha_j^2}. \tag{5.21}$$

We simplified the equation, using the condition of very low temperature, $T\ll M$. In the numerical computation below, we allow the temperature region, $T\approx M$.

Since the bulk of the cosmic medium is in thermal equilibrium, the usual temperature-time relation (3.33) holds. One can then integrate the equation for the average number density (5.17), which gives a source term Y_S to the rest of equations;

$$\begin{aligned}Y_S(\tau)&\equiv\eta^{3/4}\,\frac{n_X-n_0^{\rm th}}{T^3}\\&=(2\pi)^{-3/2}\left(e^{-\tau}\int_{\tau_i}^{\tau}dx\,x^{3/4}e^{x-\sqrt{x/\eta}}-\tau^{3/4}e^{-\sqrt{\tau/\eta}}\right)\\&\quad+\eta^{3/4}e^{-(\tau-\tau_i)}\left(\frac{n_+}{T^3}\right)_i,\end{aligned} \tag{5.22}$$

$$\frac{dY_B}{d\tau}=-\frac{K}{2}Y_0Y_B+\delta Y_- +\epsilon\,\eta^{-3/4}Y_S, \tag{5.23}$$

$$\frac{dY_-}{d\tau}=-Y_- +\frac{\tilde{\delta}}{2}Y_0\,Y_B, \tag{5.24}$$

$$Y_B\equiv\frac{n_B}{T^3},\quad Y_-\equiv\frac{n_-}{T^3},\quad Y_0\equiv\frac{n_0^{\rm th}}{T^3}. \tag{5.25}$$

Solutions to this approximation will be compared to a more precise numerical result.

A comparison with previous works [13,14] reveals some differences. In our treatment two-body processes $b_j^\dagger(\omega)|0\rangle\leftrightarrow b_k^\dagger(\omega)|0\rangle$ are not included. (In the terminology of [13] this corresponds to the off-shell two-body contribution and their on-shell two-body term is automatically included in our approach, too.) This is a higher order effect, hence was neglected in the present work. The major difference absent in the past work and included here is however the off-shell effect, as will be discussed shortly.

An improved, but a still simple approximation that incorporates the power law period of decay is to add the two contributions in the exponential and the power law periods incoherently, ignoring the interference between the pole and the power terms. This introduces a time, or temperature dependence of the rate $\Gamma(t)$ that includes the off-shell effect.

Another simplification is necessary for a practical computation of momentum integrals containing the momentum dependent effective rate and the time dilatation factor. It is not difficult to partially include the momentum dependence in quantities such as

$$\int_0^\infty\frac{dk\,k^2}{2\pi^2}\overline{\gamma}(k,t)\int_k^\infty d\omega\frac{\sigma(\omega,k)}{(\omega-\omega_k)^2+(\pi\sigma(\omega,k))^2}\frac{1}{e^{\beta\omega}-1}. \tag{5.26}$$

Indeed we did perform these momentum integrations in our numerical analysis. But unless the integro-differential equation for the occupation number, instead of the ordinary differential equation for the number density, is directly solved, it is difficult to deal with the momentum dependence in the convolution integral containing f_X and $f_{\bar{X}}$ in Eqs. (5.11). This is because these distributions are unknown before we know the complete answer to the problem, which is a formidable task. Thus, we made a replacement of the momentum factor by a temperature factor in those terms containing f_X and $f_{\bar{X}}$:

$$k^\alpha\to\frac{\Gamma(\frac{3}{2}\alpha+3)}{\Gamma(\frac{1}{2}\alpha+3)}T^\alpha, \tag{5.27}$$

using the momentum distribution function relevant at late times.

It turns out that the rest of the detailed momentum dependence integrated numerically is not crucial; its variation only changes the transient time dependence around the temperature T_* and quantities in the rest of the time region are insensitive to this momentum dependence. We therefore write down a simplified rate equation replacing the momentum dependence by its average above. In terms of the net number $Y_\pm\equiv n_\pm/T^3$ and the rescaled time $\tau\equiv\Gamma t$,

$$\frac{dY_+}{d\tau} = -\gamma(Y_+ - Y_0 - S_0), \tag{5.28a}$$

$$\frac{dY_-}{d\tau} = -\gamma\left(Y_- - \frac{\tilde{\delta}}{2}(Y_1 + S_1)Y_B \right), \tag{5.28b}$$

$$\frac{dY_B}{d\tau} = \gamma\Bigg(\epsilon Y_+ + \delta Y_- - \epsilon(Y_0 + S_0) - \frac{\delta\tilde{\delta}}{2}(Y_1 + S_1)Y_B \Bigg) - c(Y_1 + S_2)Y_B, \tag{5.28c}$$

$$\gamma = \frac{e^{-\Gamma t} + (\alpha+2)B(\alpha)\left(\frac{\Gamma}{M}\right)^{\alpha+4}\left(\frac{T}{M}\right)^{\alpha}(\Gamma t)^{-\alpha-3}}{e^{-\Gamma t} + B(\alpha)\left(\frac{\Gamma}{M}\right)^{\alpha+4}\left(\frac{T}{M}\right)^{\alpha}(\Gamma t)^{-\alpha-2}},$$

$$B(\alpha) = \frac{\Gamma(\frac{3}{2}\alpha + 3)}{\Gamma(\frac{1}{2}\alpha + 3)}, \tag{5.29}$$

$$S_0 = \frac{\zeta(\alpha+4)\Gamma(\alpha+4)\Gamma\left(\frac{\alpha}{2}+1\right)}{16\pi^2\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}+\frac{5}{2}\right)}\,\frac{\Gamma}{M}\left(\frac{T}{M}\right)^{\alpha+1}, \tag{5.30}$$

$$c = \frac{3\left(\sum_j \alpha_j^2(\gamma_j + \overline{\gamma}_j) - \frac{\delta^2}{2}\right)}{\sum_j \alpha_j^2} > 0, \tag{5.31}$$

$$S_1 = S_1(\alpha) = \frac{\zeta(\alpha+3)}{8\pi^2\sqrt{\pi}}\,\frac{\Gamma\left(\frac{\alpha}{2}+1\right)\Gamma(\alpha+3)}{\Gamma\left(\frac{\alpha}{2}+\frac{3}{2}\right)}\,\frac{\Gamma}{M}\left(\frac{T}{M}\right)^{\alpha+1}, \tag{5.32}$$

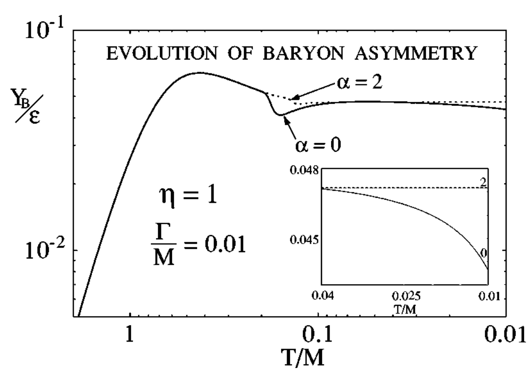

FIG. 6. Comparison of the time evolving baryon asymmetry. The case of $\alpha = 0$ shown by the solid line and enlarged in the inset gives the vanishing value for the final asymmetry, unlike the $\alpha = 2$ case shown by the dotted line.

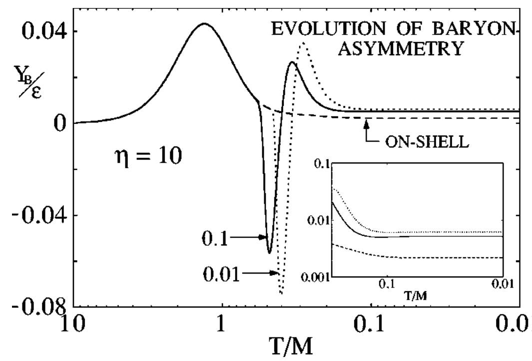

FIG. 7. Time evolution of the baryon asymmetry. Two cases of different decay rates, $\Gamma/M = 0.1, 0.01$, are compared to the evolution given by the on-shell contribution alone (the broken line). In the inset detailed behaviors are stressed.

$$S_2 = S_1(2\alpha) = \frac{\zeta(2\alpha+3)}{8\pi^2\sqrt{\pi}}\,\frac{\Gamma(\alpha+1)\Gamma(2\alpha+3)}{\Gamma(\alpha+\frac{3}{2})}\,\frac{\Gamma}{M}\left(\frac{T}{M}\right)^{2\alpha+1}, \tag{5.33}$$

$$Y_1 = \frac{1}{2\pi^2 T^2}\int_0^\infty dk\,\frac{2k^2+M^2}{\sqrt{k^2+M^2}}\,\frac{1}{e^{\sqrt{k^2+M^2}/T}-1}. \tag{5.34}$$

The low temperature approximation was not assumed here for $f^{\rm th}(\omega)$, hence

$$Y_0 = \frac{1}{2\pi^2 T^3}\int_0^\infty dk\,\frac{k^2}{e^{\sqrt{k^2+M^2}/T}-1}. \tag{5.35}$$

It can be readily proved by a rescaling argument that both Y_- and the baryon asymmetry Y_B is in direct proportion to the fundamental *CP* parameter ϵ. We assume that $\alpha \leqslant 2$ as required for any renormalizable decay interaction.

Some results of numerical integration of the time evolution equation are presented in Fig. 6 and Fig. 7. The time evolution for $\alpha = 0$ and $\alpha = 2$ is evidently different, as seen in Fig. 6. Notably, the final Y_B vanishes for $\alpha = 0$. This difference will be understandable analytically, as will be dis-

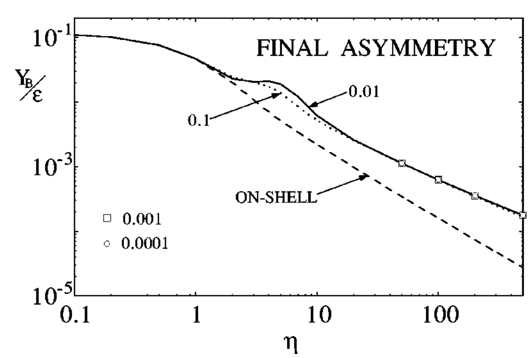

FIG. 8. Final amount of the baryon asymmetry plotted against η (= decay rate/Hubble rate at $T = M$). For comparison the result based on the on-shell contribution alone is shown by the dashed line. Those marked by open boxes and circles are results for smaller decay rates Γ/M.

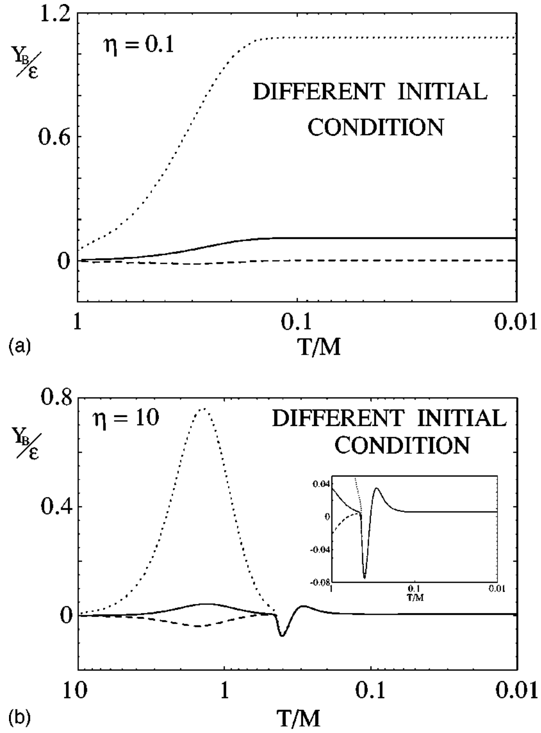

FIG. 9. Dependence of the asymmetry on the initial X abundance for different η values; 0.1 (a) and 10 (b). Result for the initially thermal abundance given by the solid line is compared to those of 10 times the thermal value (the dotted line) and the zero abundance (the dashed line).

cussed shortly. The final amount of the baryon to the photon ratio, approximately given by $Y_B = n_B/T^3$, is shown as a function of η in Fig. 8, the ratio of the decay rate to the Hubble rate at $T=M$. In all these computations of Figs. 6–8 the initial X abundance was assumed to be the thermal value. In Fig. 9 we show how the time evolution is affected by the initial condition, taking grossly different X boson abundance from the thermal value. In Figs. 6–9 we used for the parameters given by the underlying theory, $\delta = \frac{1}{3}$, $\tilde{\delta} = \frac{9}{5}$, $c=6$, chosen to be consistent with the CPT constraint.

The asymptotic behavior of solutions to this rate equation differs, depending on whether $\alpha > \frac{1}{2}$ or $\alpha \leqslant \frac{1}{2}$. First, let us assume that the baryon asymmetry Y_B approaches a constant (including 0) asymptotically as $t \to \infty$. The first and the second equation for $Y_\pm$ in the above set then gives the asymptotic solution;

$$Y_+ \to \frac{2\alpha+4}{\alpha+3} \frac{n_\infty}{T^3}, \tag{5.36}$$

$$Y_- \to \frac{3(\alpha+2)}{\alpha+3} \frac{\delta Y_B}{\sum_j \alpha_j^2} S_1. \tag{5.37}$$

Both $Y_\pm \propto (\Gamma/M)(T/M)^{\alpha+1} \propto t^{-(\alpha+1)/2}$. The asymptotic equation for Y_B differs, depending on whether $\alpha \neq 2$ or $\alpha = 2$. We shall first discuss the simpler case of $\alpha \neq 2$. The equation for the asymmetry in this case is

$$\frac{dY_B}{d\tau} \approx -c\; S_2\; Y_B, \tag{5.38}$$

where c is given by Eq. (5.31). Since $S_2 \propto \tau^{-(2\alpha+1)/2}$,

$$Y_B \approx \text{const} \times \exp\left[-c \int_{\tau_0}^{\tau} dx\; S_2(x) \right] \tag{5.39}$$

approaches a constant for $\alpha > \frac{1}{2}$. Thus, as $t \to \infty$, Y_B approaches a constant from above, its rate to the asymptote being of order $O[t^{-(2\alpha-1)/2}]$. When $\alpha=2$, the asymptotic equation for Y_B is more complicated, but the asymptotic behavior of the asymmetry is identical to the case for $2 > \alpha > \frac{1}{2}$.

On the other hand, for $\alpha \leqslant \frac{1}{2}$ the integral in the exponent of Eq. (5.39) is divergent as $\tau \to \infty$, hence $Y_B \to 0$. The asymptotic behavior is as follows: $Y_B = O[e^{-c\Gamma t^{1/2-\alpha}}]$ with $c>0$ for $\alpha < \frac{1}{2}$, and $Y_B = O[t^{-c\Gamma}]$ for $\alpha = \frac{1}{2}$. In superrenormalizable models of the boson-pair decay $\alpha=0$, and this case gives a vanishing asymptotic baryon asymmetry, unless bosons in the decay product quickly decay further into ordinary quarks and leptons.

The amount of the generated baryon asymmetry in the large η limit is of interest, because in the usual treatment of the out-of-equilibrium condition a very large mass is demanded for the X boson due to the on-shell kinematics [4]: it requires that the ratio of the decay rate to the Hubble rate at the temperature $T=M$, precisely η, should not be too large in order to get a sizable X abundance for the baryon generation. More precisely, the on-shell Boltzmann approach gives the asymmetry that depends on η like $\propto \eta^{-1.2}$ for a large η [13,14]. In Fig. 8 we observe that the off-shell effect gives a less decreasing behavior as η increases. It would be of some interest if one can work out the infinite η limit analytically.

According to the recent analysis of the reheating problem after inflation it is quite possible [15] that the heavy X boson has been created right after the explosive decay [16] of the inflaton oscillation and prior to the thermalization process. It is thus of considerable interest to examine the baryogenesis without assuming the thermal abundance of the heavy boson initially. The nonthermal initial condition has been taken in Fig. 9 and for a large enough η the final baryon asymmetry is seen insensitive to the initial condition.

In summary, we derived the cosmological time evolution equation for the abundance of unstable particles, including the off-shell effect not taken into account in the Boltzmann approach. Application to the baryogenesis problem shows that the out-of equilibrium condition based on the on-shell kinematics is changed.

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